Indian Statistical Institute, Chennai
MSTAT: Semester-I, 2016-18

Final Examination: Linear Models

Maximum marks **30**. Duration: 180 minutes

Part I. Answer all the questions

1. Discuss two way nested ANOVA model. State the assumptions so that this is a Gauss-Markov model. Represent the two way nested ANOVA model in the form $Y = X\beta + \epsilon$, when $a = 2$, $b = 2$, and $n_{ij} = 3$, where a and b are the levels of factor A and B, respectively. What is the rank of the design matrix X? [4]

2. Let $E(Y_1) = 2\beta_1 + \beta_2$, $E(Y_2) = \beta_1 - \beta_2$ and $E(Y_3) = \beta_1 + \beta_2$. Check whether the following parametric functions are estimable? If yes, obtain the BLUE of these parametric functions and find its varaince

a) $2\beta_1 - \beta_2$

b) $\beta_1 + \beta_2$

c) $\beta_1 + 2\beta_2$. [6]

3. Consider the analysis of variance model $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$, for $i = 1, 2, 3, 4$, $j = 1, 2, \ldots, n_i$ and ε_i's are independent normal variables with mean zero and variance σ^2. Construct a test for testing the hypothesis $H_0 : \mu + \alpha_1 = 5, \alpha_3 - \alpha_4 = 1$ versus $H_1 : \mu + \alpha_1 \neq 5, \alpha_3 - \alpha_4 \neq 1$. [6]

4. Consider the Gauss-Markov model $Y = X\beta + \varepsilon$, where Y is $n \times 1$ vector, X is a $n \times p$ design matrix, β is $n \times 1$ vector of regression coefficients and ε is $n \times 1$ vector of random errors and $\varepsilon \sim N_n(0, \sigma^2 I)$. Suppose that the rank

of X is $p < n$. Show that $A = X(X'X)^{-1}X'$ and $I - A = I - X(X'X)^{-1}X'$ are both symmetric and idempotent, and find the rank of each. [6]

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a standard normal distribution. Using sum of squares of X_i's, show that

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

are independently distributed. [4]

Part II. Answer either 6A or 6B

6A. Consider the analysis of covariance model

$$Y = X\beta + Z\gamma + \varepsilon,$$

where Z is a $n \times m$ matrix of rank m and γ is $m \times 1$ vector of covariates and Y, X, β and ε are as in Question 4. Construct a test for testing the null hypothesis that covariates effects are equal to zero. [6]

6B. Consider the two factor mixed model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad i = 1, 2, \ldots, r \quad \text{and} \quad j = 1, 2, \ldots, s,$$

where μ represents the overall grand mean, α_i's are constants satisfying the constraint $\sum_{i=1}^{r} \alpha_i = 0$, β_j's are independent normal variables with mean zero and variance σ_β^2 and ε_{ij}'s are independent normal variables with mean zero and variance σ^2. Construct a F test for testing the hypothesis $H_0 : \sigma_\beta^2 = 0$ versus $H_1 : \sigma_\beta^2 > 0$. [6]